5.1 Probability Distributions

\( \text{capital } X \)

\[ X = \text{ random variable } : \alpha \]

Variable whose values are determined by chance

Ex. Flip 5 pennies

\[ X = \text{ the number of heads} \]

(Classical) \[ P(X=0) = P(\text{TTTTT}) \]

\[ = \left( \frac{1}{2} \right)^5 = 0.03125 \]
Discrete Probability Distribution:

- a data display that shows all the probabilities for each value of $X$

![Histogram](image1)

<table>
<thead>
<tr>
<th>$X$</th>
<th>$P(X)$</th>
</tr>
</thead>
</table>

Probability density function

![Probability density function](image2)
Expected Value: the expected value for the random variable $X$

$E(X) = \sum (x \cdot P(X))$

- First, multiply each value of $X$ by its probability
- Then, add all the results
2. Determine whether the distribution represents a probability distribution. If it does not, state why.

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$X$</td>
<td>3</td>
<td>6</td>
<td>8</td>
<td>12</td>
</tr>
<tr>
<td>$P(X)$</td>
<td>0.3</td>
<td>0.5</td>
<td>0.7</td>
<td>$-0.8$</td>
</tr>
</tbody>
</table>

**a.** NO

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$X$</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>12</td>
</tr>
<tr>
<td>$P(X)$</td>
<td>0.3</td>
<td>0.4</td>
<td>0.2</td>
<td>0.1</td>
</tr>
</tbody>
</table>

**b.** YES
3. State whether the variable is discrete or continuous.
   a. The number of people who play the state lottery each day Discrete
   b. The blood pressures of all patients admitted to a hospital on a specific day Continuous
   c. The time it takes to have a medical physical exam Continuous
4. A die is loaded in such a way that the probabilities of getting 1, 2, 3, 4, 5, or 6 are \( \frac{1}{2}, \frac{1}{6}, \frac{1}{3}, \frac{1}{12}, \frac{1}{6}, \text{ and } \frac{1}{12} \) respectively.
   
   a. Construct a probability distribution for the data and draw a graph for the distribution.

   \[
   \begin{array}{c|ccccccc}
   X & 1 & 2 & 3 & 4 & 5 & 6 \\
   P(X) & \frac{1}{2} & \frac{1}{6} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\
   \end{array}
   \]

   Prob of rolling a particular value using a loaded die

b. The die is rolled once. What is the expected value of the roll?

\[
1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{12} + 4 \cdot \frac{1}{12} + 5 \cdot \frac{1}{12} + 6 \cdot \frac{1}{12}
\]

\[
\frac{23}{12}
\]
5. The probability that a child plays one computer game is one-half as likely as that of playing two computer games. The probability of playing three games is twice as likely as that of playing two games, and the probability of playing four games is the average of the other three. Let $X$ be the number of computer games played. Construct the probability distribution for this random variable.

\[
\begin{array}{c|cccc}
X & 1 & 2 & 3 & 4 \\
P(X) & \frac{3}{28} & \frac{3}{14} & \frac{3}{7} & \frac{1}{43.5y} \\
\end{array}
\]

\[
\frac{3}{28} + \frac{3}{14} + \frac{3}{7} + \frac{1}{43.5y} = 1
\]

\[
\frac{1}{2}y + y + 2y + \frac{3.5y}{3} = 1
\]

\[
3 \left(3.5y + \frac{3.5y}{3}\right) = (1)3
\]

\[
10.5y + 3.5y = 3
\]

\[
14y = 3
\]

\[
y = \frac{3}{14}
\]
\( \bar{X} = \text{Sample mean} \)

\( \sigma = \text{Population standard deviation} \)

\( S^2 = \text{Sample variance} \)

\( \mu = \text{Population mean} \)
5.2 Mean, Variance, and Standard Deviation

mean = average
   = expected value

\[ \mu = E(X) = \sum (X \cdot P(X)) \]

Ex. If three coins are tossed, find the mean number of heads that occur.

\[ X = \text{# of heads from 3 coins} = \{0,1,2,3\} \]

\[ \mu = E(X) = 0 \cdot P(0) + 1 \cdot P(1) + 2 \cdot P(2) + 3 \cdot P(3) \]

\[ = 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} \]

\[ = 0 + \frac{3}{8} + \frac{6}{8} + \frac{3}{8} \]

\[ = \frac{12}{8} \]

\[ = 1 \frac{1}{2} \]
Standard Deviation:

\[ \sigma = \sqrt{\sum (X^2 \cdot p(X))} - \mu^2 \]

**Ex.** Flip 3 coins. Let \( X \) = # of heads. Find the standard deviation.

\[
\begin{align*}
\sigma &= \sqrt{\left[ 0^2 \cdot p(0) + 1^2 \cdot p(1) + 2^2 \cdot p(2) + 3^2 \cdot p(3) \right] - 1.5^2} \\
&= \sqrt{\left[ 0^2 \cdot \frac{1}{8} + 1^2 \cdot \frac{3}{8} + 2^2 \cdot \frac{3}{8} + 3^2 \cdot \frac{1}{8} \right] - 1.5^2} \\
&= \sqrt{\frac{3}{8} + \frac{12}{8} + \frac{9}{8} - 2.25} \\
&= \sqrt{0.75} \\
&\approx 0.866
\end{align*}
\]

So... \( \sigma = 0.866 \) and \( \sigma^2 = 0.75 \).
1. **Defective DVDs** From past experience, a company found that in cartons of DVDs, 90% contain no defective DVDs, 5% contain one defective DVD, 3% contain two defective DVDs, and 2% contain three defective DVDs. Find the mean, variance, and standard deviation for the number of defective DVDs.

<table>
<thead>
<tr>
<th># of defect</th>
<th>X</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>P(X)</td>
<td></td>
<td>0.9</td>
<td>0.05</td>
<td>0.03</td>
<td>0.02</td>
</tr>
</tbody>
</table>

\[
\mu = \sum X \cdot P(X) = 0.17
\]

\[
\sigma^2 = 0.321
\]

\[
\sigma = 0.567
\]
2. Suit Sales  The number of suits sold per day at a retail store is shown in the table, with the corresponding probabilities. Find the mean, variance, and standard deviation of the distribution.

<table>
<thead>
<tr>
<th>Number of suits sold $X$</th>
<th>19</th>
<th>20</th>
<th>21</th>
<th>22</th>
<th>23</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability $P(X)$</td>
<td>0.2</td>
<td>0.2</td>
<td>0.3</td>
<td>0.2</td>
<td>0.1</td>
</tr>
</tbody>
</table>

If the manager of the retail store wants to be sure that he has enough suits for the next 5 days, how many should the manager purchase?

$\mu = 20.8$

$\sigma^2 = 1.56$

$\sigma = 1.25$

105
4. **Trivia Quiz**  The probabilities that a player will get 5 to 10 questions right on a trivia quiz are shown below. Find the mean, variance, and standard deviation for the distribution.

\[
\begin{array}{ccccccc}
X & 5 & 6 & 7 & 8 & 9 & 10 \\
P(X) & 0.05 & 0.2 & 0.4 & 0.1 & 0.15 & 0.1 \\
\end{array}
\]

- \( \mu = 7.4 \)
- \( \sigma^2 = 1.84 \)
- \( \sigma = 1.36 \)
5. **Cellular Phone Sales** The probability that a cellular phone company kiosk sells $X$ number of new phone contracts per day is shown below. Find the mean, variance, and standard deviation for this probability distribution.

<table>
<thead>
<tr>
<th>$X$</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X)$</td>
<td>0.4</td>
<td>0.3</td>
<td>0.1</td>
<td>0.15</td>
<td>0.05</td>
</tr>
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</table>

What is the probability that they will sell 6 or more contracts three days in a row?

0.3

0.3

0.3

\[2.7\%\]

$\mu = 5.4$

$\sigma^2 = 2.9$

$\sigma = 1.7$
State whether each of the following variables is categorical, discrete numerical, or continuous numerical.

The number of defective tires on a car
State whether each of the following variables is categorical, discrete numerical, or continuous numerical.

The body temperature of a hospital patient
State whether each of the following variables is categorical, discrete numerical, or continuous numerical.

The lifetime of a light bulb
State whether each of the following variables is categorical, discrete numerical, or continuous numerical.

The number of questions asked during a 1-hr lecture
State whether each of the following variables is categorical, discrete numerical, or continuous numerical.

The amount of water used by a household during a given month
Based on past history, a fire station reports that 25% of the calls to the station are false alarms, 60% are for small fires that can be handled by station personnel without outside assistance, and 15% are for major fires that require outside help.

Let \( x = \text{type of call} = \{\text{false alarm, small fire, and major fire}\} \)

Write 3 probability statements using the relative frequencies mentioned above.
Suppose that fund-raisers at a university call recent graduates to request donations for campus outreach programs. They report the following information for last year's graduates.

<table>
<thead>
<tr>
<th>Size of donation</th>
<th>$0</th>
<th>$10</th>
<th>$25</th>
<th>$50</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proportion of calls</td>
<td>0.45</td>
<td>0.30</td>
<td>0.20</td>
<td>0.05</td>
</tr>
</tbody>
</table>

Three attempts were made to contact each graduate; a donation of $0 was recorded both for those who were contacted but who declined to make a donation and for those who were not reached in three attempts. Consider the variable $x = \text{amount of donation}$ for the population of last year's graduates of this university.

What is the most common value of $x$ in this population?
Suppose that fund-raisers at a university call recent graduates to request donations for campus outreach programs. They report the following information for last year's graduates.

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What is $P(x \geq 25)$?
Suppose that fund-raisers at a university call recent graduates to request donations for campus outreach programs. They report the following information for last year's graduates.

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Three attempts were made to contact each graduate; a donation of $0 was recorded both for those who were contacted but who declined to make a donation and for those who were not reached in three attempts. Consider the variable \( x = \) amount of donation for the population of last year's graduates of this university.

What is \( P(x > 0) \)?
A pizza shop sells pizzas in four different sizes. The 1000 most recent orders for a single pizza gave the following proportions for the various sizes:

<table>
<thead>
<tr>
<th>Size</th>
<th>12 in.</th>
<th>14 in.</th>
<th>16 in.</th>
<th>18 in.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proportion</td>
<td>0.20</td>
<td>0.25</td>
<td>0.50</td>
<td>0.05</td>
</tr>
</tbody>
</table>

With \( x \) denoting the size of the pizza in a single-pizza order, the given table is an approximation to the population distribution of \( x \).

Approximate \( P(x < 16) \)  
Approximate \( P(x \leq 16) \)
A pizza shop sells pizzas in four different sizes. The 1000 most recent orders for a single pizza gave the following proportions for the various sizes:

<table>
<thead>
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<th>14 in.</th>
<th>16 in.</th>
<th>18 in.</th>
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<tbody>
<tr>
<td>Proportion</td>
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<td>0.25</td>
<td>0.50</td>
<td>0.05</td>
</tr>
</tbody>
</table>

With \( x \) denoting the size of the pizza in a single-pizza order, the given table is an approximation to the population distribution of \( x \).

It can be shown that the mean value of \( x \) is approximately 14.8 in. What is the approximate probability that \( x \) is within 2 in. of this mean value?
Ex. (p.270)

One thousand tickets are sold at $1 each for a color television value at $350. What is the expected value of the gain if you purchase one ticket?
Ex. (p.271)

A financial advisor suggests that his client select one of two types of bonds in which to invest $5000. Bond X pays a return of 4% and has a default rate of 2%. Bond Y has a 2.5% return and a default rate of 1%. Find the expected rate of return and decide which bond would be a better investment.

Bond X = $200 return

\[
\begin{align*}
\mathbb{E}(X) &= \frac{200}{0.98} - \frac{5000}{0.02} \\
\mathbb{E}(X) &= \$96
\end{align*}
\]

Bond Y = $125

\[
\begin{align*}
\mathbb{E}(Y) &= \frac{125}{0.99} - \frac{5000}{0.01} \\
\mathbb{E}(Y) &= \$74
\end{align*}
\]